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AN INTEGRAL EQUATION AND ITS APPLICATION TO CONTACT PROBLEMS IN THE THEORY OF ELASTICITY WITH FRICTION AND COHESION FORCES

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A singular integral equation is examined. This equation is generated by some mixed problems of the plane theory of elasticity, in particular by problems dealing with the contact between two bodies when friction or complete cohesion are present in the contact region. General properties of the solution of this equation are investigated. The initial singular equation is reduced to Fredholm's integral equation of the second kind through application of regularization by means of the solution of the characteristic equation [1]. For the condition where the kernel is small the resolvent is found for Fredholm's integral equation of the second kind.

Problems of interaction between a stamp and an elastic isotropic strip are examined: displacement of the stamp in the presence of friction between the stamp and the strip, and the impression of the stamp into the strip in case of complete cohesion in the region of contact (*). Solutions of these problems are obtained in the form of power series of a dimensionless small parameter which characterizes the relative length of the contact region. Boundaries for uniform and absolute convergence of these series are established. Examples are presented.

1. Let us examine the following singular integral equation:

$$\frac{\theta}{2}\int_{-1}^{1}\varphi(\xi)\operatorname{sgn}(x-\xi)\,d\xi-\frac{1}{\pi}\int_{-1}^{1}\varphi(\xi)\ln\mu\,|x-\xi|\,d\xi=\vartheta(x,\mu),\quad |x|\leqslant 1 \tag{1.1}$$

*) Analogous problems on interaction of a stamp with an elastic half-plane were examined in a number of papers by other authors (see, for example, appropriate problems and their reviews in [2]).

where

$$\vartheta(x,\mu) = -\delta f(x) - \frac{1}{\pi}\int_{-1}^{1}\varphi(\xi)k(x,\xi,\mu)\,d\xi \tag{1.2}$$

Here $\varphi(x)$ is the desired function, $\delta f(x)$ and $k(x, \xi, \mu)$ are given functions. Let us assume that the second derivative of the function $k(x, \xi, \mu)$ with respect to variable x is bounded in the rectangle $\{-1 \leqslant x \leqslant 1,\ -1 \leqslant \xi \leqslant 1\}$ for all values of parameter $\mu \in [0, \infty)$. It will also be assumed that in the general case the functions $\varphi(x)$, $f(x)$, $k(x, \xi, \mu)$, and the parameter θ are complex (*).

Some plane mixed problems of the theory of elasticity, for example contact problems of two bodies, are reduced to integral equation (1.1). In this connection the case $\theta = 0$ corresponds to the absence of friction in the region of contact; $\theta = k\Delta$ indicates the presence of friction forces in the contact region, and $\theta = i\Delta$ corresponds to complete cohesion in the contact region. Here k is the friction coefficient, $\Delta = (1-2\nu)/2(1-\nu)$, and ν is Poisson's ratio. The case $\vartheta(x,\mu) \equiv \delta f(x)$ corresponds to the problem of an absolutely rigid stamp being impressed into an elastic half-plane under conditions of contact examined above.

Differentiating both parts of Eq. (1.1) with respect to τ, we arrive at a singular integral equation with constant coefficients. The equation has the form (47.5) of [1]. The solution of this equation is given by expressions (47.12) and (47.13) of the same reference. For this case the solution takes the form

$$\varphi(x) = (4g)^{-1}(g+1)^2 X(x)\,\psi(x,\mu), \qquad |x| \leqslant 1 \tag{1.3}$$

where

$$\psi(x,\mu) = C - \left[\frac{1}{\pi}\int_{-1}^{1}\frac{\vartheta'(\xi,\mu)\,d\xi}{X(\xi)(\xi - x)} - \theta\frac{\vartheta'(x,\mu)}{X(x)}\right] = C + \delta\vartheta_0(x) +$$

$$+\frac{(g+1)^2}{4\pi g}\int_{-1}^{1}X(\xi)\psi(\xi,\mu)\left[\frac{1}{\pi}\int_{-1}^{1}\frac{k_t'(t,\xi,\mu)}{X(t)(t-x)} - \theta\frac{k_x'(x,\xi,\mu)}{X(x)}\right]d\xi$$

$$\vartheta_0(x) = \frac{1}{\pi}\int_{-1}^{1}\frac{f'(\xi)\,d\xi}{X(\xi)(\xi - x)} - \theta\frac{f'(x)}{X(x)}, \quad g = \frac{1+i\theta}{1-i\theta} \tag{1.4}$$

$$X(x) = (1-x)^{-1/2+\omega}(1+x)^{-1/2-\omega}, \quad \omega = (2\pi i)^{-1}\lg g \tag{1.5}$$

(C is an arbitrary constant).

It is convenient to introduce here the following integrals which will be needed later. These integrals are readily computed by the method proposed by Muskhelishvili [2] (Sect. 110, Chapter VI, Note 1)

$$\frac{1}{\pi}\int_{-1}^{1}\frac{t^n\,dt}{X(t)(t-x)} = \theta\frac{x^n}{X(x)} - \frac{2\sqrt{g}}{g+1}P_{n+1}(x)$$

$$P_n(x) = \sum_{k=0}^{n}\gamma_{n-k}x^k, \quad P_{-n-1}(x) \equiv 0 \quad (n = 0, 1, 2, \ldots) \tag{1.6}$$

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*) Without loss of generality this treatment was limited to the case (1.2) in order to abbreviate the writing of equations. However, the regular integral term in the right side of Eq. (1.1) can contain real and imaginary parts of function $\varphi(x)$, i.e.

$$\vartheta(x,\mu) = -\delta f(x) - \frac{1}{\pi}\int_{-1}^{1}[k(x,\xi,\mu)\varphi(\xi) + k_1(x,\xi,\mu)\mathrm{Re}\,\varphi(\xi) + k_2(x,\xi,\mu)\,\mathrm{Im}\,\varphi(\xi)]\,d\xi$$

$$\gamma_n = \frac{1}{n!}\sum_{j=0}^{n}(-1)^j C_n^j \prod_{p=0}^{j-1}\prod_{q=0}^{n-j-1}\left(\frac{1}{2}-\omega-p\right)\left(\frac{1}{2}+\omega-q\right)$$

$$\frac{1}{\pi}\int_{-1}^{1}\frac{t^{n+1}X(t)\,dt}{t-x} = -\theta x^{n+1}X(x) + \frac{2\sqrt{g}}{g+1}S_n(x)$$

$$S_n(x) = \sum_{k=0}^{n}\delta_{n-k}x^k, \quad S_{-n-1}(x) \equiv 0 \quad (n = 0, 1, 2, \ldots) \tag{1.7}$$

$$\delta_n = \frac{1}{n!}\sum_{j=0}^{n}(-1)^{n-j}C_n^j \prod_{p=0}^{n-j}\prod_{q=0}^{n-j-1}\left(\frac{1}{2}-\omega+p\right)\left(\frac{1}{2}+\omega+q\right)$$

In the case where the upper limit of the product is equal (−1), the product is taken equal (+1).

Utilizing Eq. (1.6) for $n = 0$, we can write expression (1.4) in the form

$$\psi(x, \mu) = C + 2\sqrt{g}(g+1)^{-1}P_1(x)\vartheta'(x, \mu) - J(x, \mu) \tag{1.8}$$

$$J(x, \mu) = \frac{1}{\pi}\int_{-1}^{1}\frac{F(x, \xi, \mu)\,d\xi}{X(\xi)}, \quad F(x, \xi, \mu) = \frac{\vartheta'(\xi, \mu) - \vartheta'(x, \mu)}{\xi - x} \tag{1.9}$$

A theorem is given below which establishes some general properties of the solution of the integral equation (1.1) as a function of properties of the right side.

Theorem 1.1. If function $\vartheta(x, \mu) \in H_n^\lambda(-1,1)$ for any fixed value of parameter $\mu \in [0, \infty)$, then the function $\psi(x, \mu) \in C^{(n-1)}(-1,1)$ for all $\mu[0, \infty]$ (*).

In Eqs. (1.8) and (1.9) let us fix the arbitrary values of parameter μ and it will be omitted in the proof of the theorem.

It is apparent that the statement will be proven if it is shown that $J(x) \in C^{n-1}(-1, 1)$.

Differentiating the integral (1.9) with respect to x $(n-1)$ times and utilizing Eq. (7.4) of [3], we obtain

$$J^{(n-1)}(x) = \frac{1}{\pi}\int_{-1}^{1}\frac{N(\xi, x)\,d\xi}{X(\xi)\,|\xi - x|^\sigma} \tag{1.10}$$

where the function $N(\xi, x)$ satisfies Hölder's condition with respect to both variables, while σ is an arbitrary value contained in the interval $1 - \lambda < \sigma < 1$.

Taking into account that the function under the integral $X^{-1}(\xi)\,N(\xi, x)$ is bounded with respect to both variables, the uniform convergence of integral (1.10) is not difficult to establish. Consequently, $J^{(n-1)}(x) \in C(-1,1)$, which proves the theorem.

Let us now turn to integral equation (1.1).

Let $f(x) \in H_1^\lambda(-1,1)$. By virtue of assumptions made earlier with respect to function $k(x, \xi, \mu)$ we obtain on the basis of Theorem 1 that the function $\psi(x, \mu) \in C(-1, 1)$ and the operator

$$Lf \equiv \frac{(g+1)^2}{4\pi g}\int_{-1}^{1}X(\xi)f(\xi)\left[\frac{1}{\pi}\int_{-1}^{1}\frac{k_t'(t, \xi, \mu)\,dt}{X(t)(t-x)} - \theta\,\frac{k_x'(x, \xi, \mu)}{X(x)}\right]d\xi \tag{1.11}$$

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*) Here $H_n^\lambda(-1,1)$ designates a class of functions, the nth derivative of which satisfies the Hölder condition on the segment $x \in [-1,1]$ with the index λ. $C^{(n)}(-1,1)$ is a class of functions which on the same segment have a continuous nth derivative.

in expression (1.4) operates in C (−1, 1). In this connection the singular equation (1.1) is reduced to Fredholm's integral equation of the second kind (1.4) with respect to a new unknown function $\psi(x, \mu)$. On the basis of Theorem 5 (Section 5, Chapter IV) of [4] we obtain that in the case when $\|L\| < 1$, the solution of Eq. (1.4) assumes the form (*)

$$\psi = \sum_{k=0}^{\infty} L^k \psi_0, \qquad \psi_0(x) = C + \delta\vartheta_0(x) \tag{1.12}$$

and series (1.12) converges uniformly and absolutely for all values of parameter μ, which satisfy the following inequality

$$\|L\| \leqslant |1 + 2\omega| \{ \max |k_x'(x, \xi, \mu)| + (x, \xi \in [-1,1]) + \\ + {}^1/_2 |1 - 2\omega| \max |k_{xx}''(x, \xi, \mu)| \} < 1 \tag{1.13}$$

Let us examine some problems on interaction of a stamp with an isotropic strip of the width h for the case where planar deformation occurs (transition to the case of plane state of stress is accomplished according to known equations).

Let $2a$ be the length of the contact region. Dimensionless coordinates (x', y') are introduced according to equations $x = ax'$, $y = hy'$ such that the strip occupies the region $\{-\infty < x' < \infty, -1 \leqslant y' \leqslant 0\}$ and the origin of the coordinate system coincides with the center of the contact region: $-1 \leqslant x' \leqslant 1$ ($2\mu = 2a / h$ is the relative length of th contact region). The problem will be examined below in dimensionless coordinates (x', y') with omission of primes.

2. Let us examine the problem of equilibrium of the stamp on the boundary of the strip in the presence of friction in the contact region. Let us assume that under the stamp $\tau(x) = -kq(x)$, where $q(x)$ and $\tau(x)$ denote distribution functions of normal and tangential stresses, respectively, developed under the stamp (k is the friction coefficient, assumed to be constant). Let us assume further that additional loading is absent outside the stamp and that the opposite boundary of the strip rests on a nondeformable base with the following conditions: (a) absence of friction between the base and strip; (b) complete cohesion of boundary points.

Through methods of operational calculus using the Fourier transform, problems (a) and (b) are reduced to the determination of the unknown distribution function of normal stresses $q(x)$ in the contact region from the following integral equation:

$$\int_{-1}^{1} q(\xi) K[\mu(x-\xi)]\, d\xi = -\delta f(x), \quad |x| \leqslant 1 \tag{2.1}$$

where

$$K(\mu t) = \frac{1}{\pi} \int_0^{\infty} [L_1(u) \cos \mu t u + k\Delta L_2(u) \sin \mu t u] \frac{du}{u} \tag{2.2}$$

$$\text{(a)} \quad L_1(u) = \frac{\operatorname{sh}^2 u}{\operatorname{ch} u \operatorname{sh} u + u}, \quad L_2(u) = \frac{\operatorname{ch} u \operatorname{sh} u - (1-2\nu)^{-1} u}{\operatorname{ch} u \operatorname{sh} u + u}$$

$$\text{(b)} \quad L_1(u) = \frac{\varkappa \operatorname{ch} u \operatorname{sh} u - u}{\varkappa \operatorname{ch}^2 u + u^2 + (1-2\nu)^2}, \quad L_2(u) = \frac{\varkappa \operatorname{sh}^2 u - (1-2\nu)^{-1} u^2}{\varkappa \operatorname{ch}^2 u + u^2 + (1-2\nu)^2} \tag{2.3}$$

$$\Delta = (1-2\nu) / 2(1-\nu), \quad \varkappa = 3 - 4\nu, \quad \delta = E / 2(1-\nu^2)$$

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*) The operator L^k denotes successive application of operator L (1.11). For example, $L(L\psi_0) = L^2\psi_0$.

Here $f(x)$ is the settling function of boundary points of the strip under the stamp, E is Young's modulus, and ν is Poisson's ratio.

Let us separate the characteristic part from kernel $K(t)$ of the integral equation (2.1). We obtain

$$K(\mu t) = 2^{-1} k\Delta \operatorname{sgn} t - \pi^{-1}[\ln \mu |t| - k(\mu t)] \quad (t = x - \xi) \tag{2.4}$$

The function

$$k(\sigma) = \int_0^\infty \{[L_1(u) - 1]\cos \sigma u + e^{-u} + k\Delta [L_2(u) - 1]\sin \sigma u\} \frac{du}{u} \tag{2.5}$$

is continuous together with all its derivatives with respect to σ on the segment $\sigma \in [-2\mu, 2\mu]$ for any fixed value of parameter $\mu \in [0, \infty)$.

Substituting expression (2.4) into Eq. (2.1), we arrive at an integral equation of the form (1.1), where $\theta = k\Delta$ and $k(x, \xi, \mu) = k[\mu(x - \xi)]$. Regularization of the latter by means of the solution of the characteristic equation leads to Fredholm's equation (1.4). It follows from (1.3)–(1.5) that the solution of Eq. (2.1) can be represented in the form

$$q(x) = [1 + (k\Delta)^2]^{-1} X(x)\psi(x, \mu), \quad \psi(x, \mu) = \psi_0(x) + \psi^1(x, \mu)$$
$$X(x) = (1 - x)^{-1/2+\alpha}(1 + x)^{1/2-\alpha}, \quad \alpha = \pi^{-1} \operatorname{arctg} k\Delta \tag{2.6}$$

where $\psi_0(x)$ is the solution of the corresponding problem for the half-plane. The term $\psi^1(x, \mu)$ is due to the presence of boundary of the strip on the opposite side from the stamp. It follows also from (1.4) that the smaller the parameter μ (the wider the strip), the smaller the contribution of this term. From properties of function (2.5) on the basis of Theorem 1.1 we find that $\psi^1(x, \mu) \in C^{(\infty)}(-1, 1)$ for all $\mu \in [0, \infty)$. Equation (1.12) gives asymptotic representations of function $\psi(x, \mu)$ for small values of parameter μ. In this connection we find from (1.13) that the series (1.12) converges uniformly and absolutely for

$$\mu < \mu_0, \ \mu_0 = \frac{-\max|k'(\sigma)| + R(\sigma; \alpha)}{(1 - 2\alpha)\max|k''(\sigma)|} \quad (-\infty < \sigma < +\infty) \tag{2.7}$$

$$R(\sigma; \alpha) = [(\max|k'(\sigma)|)^2 + 2(1-2\alpha)(1+2\alpha)^{-1}\max|k''(\sigma)|]^{1/2}$$

For calculation of quadratures in expression (1.12) the function $k(\sigma)$ is expanded in Maclaurin series. We obtain

$$k(\sigma) = \sum_{s=0}^\infty a_s \sigma^s = \sum_{s=0}^\infty a_s \mu^s y^s \quad (-2 \leqslant y = x - \xi \leqslant 2) \tag{2.8}$$

$$a_0 = \int_0^\infty [L_1(u) - 1 + e^{-u}]\,du, \quad a_{2m-1} = k\Delta \frac{(-1)^{m-1}}{(2m-1)!}\int_0^\infty [L_2(u) - 1]\,u^{2m-2}\,du$$

$$a_{2m} = \frac{(-1)^m}{(2m)!}\int_0^\infty [L_1(u) - 1]\,u^{2m-1}\,du \quad (m = 1, 2, \ldots) \tag{2.9}$$

The series (2.8) converges uniformly and absolutely for all $\mu < 1$.

Now substituting (2.8) into expression (1.12), computing quadratures from Eqs. (1.6) and (1.7), and collecting coefficients for identical powers of parameter μ, we obtain that the solution of integral equation (2.1) for small values of parameter μ has the form (2.6), where

$$\psi(x, \mu) = \sum_{j=0}^\infty \mu^j \psi_j(x), \quad \psi_0(x) = C + \delta\vartheta_0(x) \tag{2.10}$$

$$\psi_j(x) = \sum_{s=1}^j s a_s \sum_{r=1}^s (-1)^r C_{s-1}^{r-1} \gamma_{r-1, j-s} P_{s-r+1}(x) \quad (j \geqslant 1) \tag{2.11}$$

Coefficients $\gamma_{n,j}$ are determined from recursion relation

$$\gamma_{r,j} = \sum_{s=1}^{j} s a_s \sum_{l=1}^{s} (-1)^l C_{s-1}^{l-1} \gamma_{l-1,j-s} d_{r,s-l+1}, \quad d_{r,s} = \sum_{p=0}^{s} \gamma_{s-p} \delta_{r+p} \tag{2.12}$$

$$\gamma_{r,0} = C\delta_r + \delta \frac{g+1}{2\pi\sqrt{g}} \int_{-1}^{1} \xi^r X(\xi) \left[\frac{1}{\pi} \int_{-1}^{1} \frac{f'(t)\,dt}{X(t)(t-\xi)} - \theta \frac{f'(\xi)}{X(\xi)} \right] d\xi$$

Here the series (2.10) will converge uniformly and absolutely for all $\mu < \mu_1 = \inf\{1, \mu_0\}$.

The constant C is determined from the given force Q which impresses the stamp using the following equation:

$$Q = \int_{-1}^{1} q(x)\,dx = \frac{\pi}{\sqrt{1+(k\Delta)^2}} \sum_{j=0}^{\infty} \mu^j \gamma_{0,j} \tag{2.13}$$

In the case where $f(x) \equiv -A_n x^n$ $(n = 0,1,2,\ldots)$ we will have for $\gamma_{n,0}$ in equations (2.11)

$$\gamma_{r,0} = C\delta_r + \delta n A_n \sqrt{1+(k\Delta)^2} \sum_{p=0}^{n} \gamma_{n-p} \delta_{r+p}$$
$$\psi_0(x) = C + \delta n A_n \sqrt{1+(k\Delta)^2}\, P_n(x) \tag{2.14}$$

For $k = 0$ (then $\alpha = 0$) we obtain the solution for the ideal case where the friction is absent [5].

3. Let us examine the problem of impressing the stamp into an elastic strip in the case of complete cohesion of boundary points of the strip in the region of contact with the base of the stamp. It is also assumed that there is no additional loading outside the stamp and that the opposite boundary of the strip is in condition (a) or (b) of Sect. 2.

In the same manner as above, these problems are reduced by methods of operational calculus to the determination of normal $q(x)$ and tangential $\tau(x)$ stresses, which develop under the stamp, from the singular integral equation

$$\int_{-1}^{1} \varphi(\xi) K[\mu(x-\xi)]\,d\xi + \frac{i}{\pi} \int_{-1}^{1} \operatorname{Im}\varphi(\xi) k_1[\mu(x-\xi)]\,d\xi = -\delta f(x) \tag{3.1}$$

$$\varphi(x) = q(x) + i\tau(x), \quad |x| \leqslant 1$$

$$k_1(\sigma) = \int_{0}^{\infty} L_3(u) \cos \sigma u\,du$$

$$\text{(a)} \quad L_3(u) = -(\operatorname{ch} u \operatorname{sh} u + u)^{-1} \tag{3.2}$$
$$\text{(b)} \quad L_3(u) = 2[\varkappa \operatorname{ch}^2 u + u^2 + (1-2\nu)^2]^{-1}$$

Here the function $K(\mu, t)$ $(t = x - \xi)$ has the form of (2.2) and (2.3); in this case in Eq. (2.2) the constant $k = i$ the imaginary unit; the remaining notations have the same meaning as in the previous case.

Separation of the characteristic part in the integral equation (3.1) leads us to an equation of the form (1.1) in which $\theta = i\Delta$ and

$$\vartheta(x, \mu) = -\delta f(x) - \frac{1}{\pi}\left\{ \int_{-1}^{1} \varphi(\xi) k[\mu(x-\xi)]\,d\xi + i \int_{-1}^{1} \operatorname{Im}\varphi(\xi) k_1[\mu(x-\xi)]\,d\xi \right\} \tag{3.3}$$

Here $k(\sigma)$ $[\sigma = \mu(x - \xi)]$ in the first integral coincides with (2.5), where it is appropriate to write $k = i$.

After regularization of this equation we find that its solution has the form

$$\varphi(x) = (4\varkappa)^{-1}(\varkappa + 1)^2 X(x)\psi(x, \mu) \tag{3.4}$$
$$X(x) = (1 - x)^{-1/2+i\beta}(1 + x)^{-1/2-i\beta}, \quad \beta = (2\pi)^{-1}\ln\varkappa$$

In analogy to previous development it is possible to show that the function $\psi(x, \mu) = \psi_0(x) + \psi^1(x, \mu)$, where $\psi_0(x)$ is the solution of the corresponding problem for the half-plane, and $\psi^1(x, \mu) \in C^\infty(-1,1)$, for any fixed $\mu \in [0, \infty)$ and when $\mu \to 0$, approaches zero uniformly with respect to x. For small values of parameter μ the function $\psi(x, \mu)$ has the form (1.12), where

$$Lf \equiv \mu\frac{(\varkappa+1)^2}{4\pi\varkappa}\left\{\int_{-1}^{1} X(\xi)f(\xi)\left[\frac{1}{\pi}\int_{-1}^{1}\frac{k'_1[\mu(t-\xi)]\,dt}{X(t)(t-x)} - i\Delta\frac{k'[\mu(x-\xi)]}{X(x)}\right]d\xi + \right.$$
$$\left. + i\int_{-1}^{1} X(\xi)\operatorname{Im} f(\xi)\left[\frac{1}{\pi}\int_{-1}^{1}\frac{k'[\mu(t-\xi)]dt}{X(t)(t-x)} - i\Delta\frac{k_1'[\mu(t-\xi)]}{X(x)}\right]d\xi\right\} \tag{3.5}$$

whereby we find from (1.13) that the series (1.12) converges uniformly and absolutely for

$$\mu < \mu_0, \qquad \mu_0 = \frac{-\max|\omega(\sigma)| + \Omega(\sigma)}{\sqrt{1+4\beta^2}\max|\omega'(\sigma)|}$$
$$\Omega(\sigma) = [(\max|\omega(\sigma)|)^2 + 2\max|\omega'(\sigma)|]^{1/2}, \tag{3.6}$$
$$\omega(\sigma) = k'(\sigma) + k_1'(\sigma)$$

In the operator L let us now expand the functions $k(\sigma)$ and $k_1(\sigma)$ into Maclaurin series. In the expression for $\psi(x, \mu)$ we collect coefficients for the same powers of parameter μ. We obtain that $\psi(x, \mu)$ has the form (2.10). The coefficients $\psi_j(x)$ are determined from relationships

$$\psi_0(x) = C + \delta\vartheta_0(x)$$
$$\psi_j(x) = \sum_{s=1}^{j}\sum_{r=1}^{s}(-1)^r C_{s-1}{}^{r-1}s\,[a_s\gamma_{r-1,\,j-s} + ib_s\operatorname{Im}\gamma_{r-1,\,j-s}]\,P_{s-r+1}(x) \tag{3.7}$$
$$\gamma_{r,j} = \sum_{s=1}^{j}\sum_{l=1}^{s}(-1)^l C_{s-1}{}^{l-1}s\,[a_s\gamma_{l-1,\,j-s} + ib_s\operatorname{Im}\gamma_{l-1,\,j-s}]\,d_{r,s-l+1}$$
$$d_{r,s} = \sum_{p=0}^{s}\gamma_{s-p}\delta_{r+p} \qquad \left(C_n{}^m = \frac{n!}{m!\,(n-m)!}\right)$$

Here the initially given $\gamma_{r,0}$ $(r = 0, 1, 2, \ldots)$ coincide with (2.11). Coefficients a_s are given by relationships (2.9) where the constant k should be replaced by the imaginary unit i, and the constants b_s have the form

$$b_s = 0 \quad (s = 2m+1)$$
$$b_s = \frac{(-1)^m}{(2m)!}\int_0^\infty L_3(u)\,u^{2m}du \quad (s = 2m) \tag{3.8}$$
$$(m = 0, 1, 2, \ldots)$$

In this case the series (2.10) converges uniformly and absolutely for all $\mu < \mu_1 = \inf\{\mu_0, 1\}$; here μ_0 is found from Eq. (3.6). The constant C is determined by giving the stresses which impress the stamp into the strip, using one of the following conditions

$$Q + iT = \int_{-1}^{1} \varphi(\xi)\, d\xi = \pi \frac{\varkappa + 1}{2\sqrt{\varkappa}} \sum_{j=0}^{\infty} \mu^j \gamma_{0,j}$$

$$M = -\operatorname{Re} \int_{-1}^{1} \xi\varphi(\xi)\, d\xi = -\pi \frac{\varkappa + 1}{2\sqrt{\varkappa}} \sum_{j=0}^{\infty} \mu^j \operatorname{Re} \gamma_{1,j} \tag{3.9}$$

Here Q is the force of impression, T is the displacement force, M is the moment acting on the stamp.

In the case where $f(x) \equiv -A_n x^n \quad (n = 0, 1, 2, \ldots)$,

$$\psi_0(x) = C + 2\delta n A_n \sqrt{\varkappa}(\varkappa + 1)^{-1} P_n(x) \tag{3.10}$$

$$\gamma_{r,0} = C\delta_r + 2\delta n A_n \sqrt{\varkappa}(\varkappa + 1)^{-1} \sum_{k=0}^{n} \gamma_{n-k}\delta_{r+k} \quad (r = 0, 1, 2, \ldots)$$

Let the stamp with a rectilinear base, located on the boundary of the elastic strip, have only vertical displacement under the action of the force Q. In this case $f(x) \equiv \equiv \text{const}$ $(n = 0)$. Limiting ourselves in Eqs. (3.7) and (3.10) to terms of the order of μ^2, we have

$$\varphi(x) = (2\pi\sqrt{\varkappa})^{-1} Q X(x) [1 + \mu a_1^\circ (2\beta - ix) + O(\mu^2)] \tag{3.11}$$

Here

$$a_1^\circ = -ia_1 = \Delta \int_0^{\infty} [L_2(u) - 1]\, du \tag{3.12}$$

Separating the real and imaginary parts, we find the distribution functions of normal and tangential forces under the stamp

$$q(x) = \frac{Q}{2\pi\sqrt{1 - x^2}} \frac{\varkappa + 1}{\sqrt{\varkappa}} \{\cos\lambda(x) + \\ + \mu a_1^\circ [2\beta\cos\lambda(x) + x\sin\lambda(x)] + O(\mu^2)\} \tag{3.13}$$

$$\tau(x) = \frac{Q}{2\pi\sqrt{1 - x^2}} \frac{x + 1}{\sqrt{\varkappa}} \{\sin\lambda(x) + \mu a_1^\circ [2\beta\sin\lambda(x) - x\cos\lambda x] + O(\mu^2)\}$$

$$\lambda(x) = \beta \ln [(1 + x)(1 - x)^{-1}] \tag{3.14}$$

For $\mu \to 0$ Eqs. (3.13) agree with Eqs. (7) and (8), Sect. 114a in [2] (if in the latter one takes into account a factor of $1/4$ which was omitted).

It follows from the above equations that when approaching the boundaries of contact $x = \pm 1$, the functions $q(x)$ and $\tau(x)$ change sign. In this connection the distribution of points of changing sign depends on Poisson's ratio ν and the relative width of the contact region μ. Similar behavior was also noted for corresponding axisymmetric problems [6].

Finally, let us examine the case of an inclined stamp with a rectilinear base. The stamp penetrates in such a manner that the principal vector of external forces acting on the stamp is equal to zero. The base of stamp forms an angle ε with the axis Ox counting from Ox in the negative direction. In this case $f(x) = -\varepsilon x$, $Q + iT = 0$ and limiting ourselves to terms of the order of μ^3 we find from (3.9), (3.10) and (3.4)

$$\varphi(x) = 2\pi G\varepsilon\varkappa^{-1/2} [1 + \mu^2 a_2(1 + 4\beta^2) + O(\mu^3)] X(x) P_1(x) \tag{3.15}$$

$$M = -2\pi G\varepsilon(1 + \varkappa)^{-1}(1 + 4\beta^2)[1 + \mu^2 a_2(1 + 4\beta^2) + O(\mu^3)] \tag{3.16}$$

After separation of imaginary and real parts in expression (3.15) we obtain the formulas for distribution of normal and tangential forces under the stamp

$$q(x) = 2\ G\varepsilon\varkappa^{-1/2}\ [1 + \mu^2 a_2(1 + 4\beta^2) + O(\mu^3)](1 - x^2)^{-1/2}\ [x\ \cos\ \lambda(x) - 2\beta\ \sin\ \lambda(x)] \tag{3.17}$$

$$\tau(x) = 2\,G\,\varepsilon\varkappa^{-1/2}[1 + \mu^2 a_2(1 + 4\beta^2) + O(\mu^3)]\ (1 - x^2)^{-1/2}\,[x \sin \lambda\ (x) + 2\beta \cos \lambda\ (x)]$$

Equation (3.16) gives the relationship between moment M acting on the stamp and the angle of rotation ε.

Let us bring out the following fact which can be easily seen from relationships (3.7) and (3.9): if for any shape of the base a load acts on the stamp for which the principal force vector is equal to zero, we obtain from indicated relationships that $\gamma_{0,j} = 0$ and $\psi_1(x) \equiv 0$. This means that the refinement of the solution for the corresponding problem in a half-plane has the order of μ^2.

We note that approximate solutions of integral equations (2.1) and (3.1) for large values of parameter μ can be obtained using results in [7]. In this connection, taking into account the behavior of the solution at the ends of the segment $x \in [-1,1]$, the function under the integral $u^{-1}L\,(u) = u^{-1}\,[L_1(u) + c\Delta L_2(u)]$
($c = -ik$ in the case (2.1) and $c = 1$ in the case (3.1))
in the kernel of these equations should be approximated by the following expression

$$(A + Bu)^{1/2-\omega}\,(C + Du)^{1/2+\omega}\,Q\,(u)\,P^{-1}(u) \tag{3.18}$$

in which the coefficients A, B and C, D and the entire functions $Q(u)$ and $P(u)$ are selected on the basis of best approximation. Such a construction ensures elementary factorization of the above indicated function and also the necessary singularities of the solution at the ends $x = \pm 1$. The author proposes to present this solution and also a numerical analysis of the combination of both solutions for large and small values of the parameter μ in the next paper.

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